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A Note on the Uncertainty Analysis of Space-Time Prisms based on the Moment-Design Method

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Abstract

Space-time prism (STP) is a key concept in geography that measures the movement of objects in space and time. A space-time prism can be treated as a result of the potential path line revolving around in the three dimensional space. Though the concept has found applications in time geography, research on the analysis of uncertainty in space-time prism, particularly under high degree of nonlinearity, is scanty. Based on the moment-design (M-D) method, this paper proposes an approach to deal with nonlinear error propagation problems on the potential path areas and their intersections. In comparison with the Monte Carlo method and the implicit function method, simulation results show the advantages of the M-D method for the analysis of error propagation in space-time prisms.

Keywords: error propagation, space-time prism, spatial path area, moment-design.

1. Introduction

Space-time prism is a concept central to the study of spatio-temporal movements of objects in time geography. The classic space-time prism (STP) model deals with the set of all space-time points that can be reached by an individual from origin to destination given a maximum speed. In other words, STP is a possible location set in which each element can be represented as a three dimensional vector (x, y, t) in space and time with the first two components being the spatial coordinates, and the third component being the time coordinate. We can not only use STP to model accessibility (Miller, 1991; Kim and Kwan, 2003), but can also use it to manage the uncertainty of positions of the moving objects on a road network over time (Kuijpers and Othman, 2009; Kuijpers *et al*. 2010). Moreover, Ettema and Timmermans (2007) analyzed the uncertainty on the travel time, Miller (2005a-b), and Miller and Bridwell (2009) described the uncertainty from the perspective of travel speed, Kobayashi *et al*. (2011) proposed a simple geometric framework to study the issue of error propagation in STPs, Neutens *et al*. (2007), and Delafontaine *et al*. (2011) also tried to explain the uncertainty on STP by the rough set theory. However, there is very little research in the existing literature on the relaxation of independence between spatial parameters and temporal variables. A general theory on uncertainty in time geography has yet to be constructed.

Error analysis has been playing a very important role in data acquisition and processing. Uncertainty propagation in geographical information systems (GIS) has already been intensively investigated in GIS science (Heuvelink, 1998; Zhang and Goodchild, 2002; Leung *et al*., 2004a-d). The most common procedures for handling uncertainty are Taylor series expansion and Monte Carlo simulation. The former is an analytical and linearized method while the latter is a process of stochastic simulation. However, the limitations of Taylor series expansion and the large-sample requirement of Monte Carlo simulation restrict their applications. It is especially constraining when the transfer function is nonlinear. The M-D method (Zhang, 2006) is mainly inspired by the fact that the calculating error or precision of the formulae of error propagation is related to the number of maximum order of the moments of the noise variables. The main idea of this paper is to solve uncertainty propagation on the potential path areas (PPA) and their intersections by the M-D method. Simulation results of error propagation obtained by the M-D method, the Monte Carlo method and the implicit function method of Kobayashi *et al*. (2011) are then compared for assessment.

2. Error Propagation in STP based on the M-D Method

Let $\mathbf{X}$ be a *s*-dimensional vector which is either discrete or continuous with mean $\boldsymbol{\mu}_x \equiv E(\mathbf{X})$ and covariance matrix $\boldsymbol{\Sigma}_x \equiv \mathrm{Cov}(\mathbf{X})$. Let $\hat{\mathbf{X}}$ be a discrete random vector that approximates $\mathbf{X}$, with mean and covariance matrix equal to that of the original vector $\mathbf{X}$, i.e.

$$\boldsymbol{\mu}_{\hat{x}} = \boldsymbol{\mu}_x \ , \tag{2.1}$$

$$\boldsymbol{\Sigma}_{\hat{x}} = \boldsymbol{\Sigma}_x \ . \tag{2.2}$$

Let the discrete probability distribution of the random vector $\hat{\mathbf{X}}$ be provided in advance. Assume that $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m\}$ is a set of samples formed by all possible values of $\hat{\mathbf{X}}$, where *m* is the number of its realizations. Thus, each element in the set is a *s*-dimensional vector. Let the set of the corresponding probability values be denoted as $\{p_1, p_2, ..., p_m\}$, with each being a rational number between zero and one. By solving the system of equations in (2.2), the samples of $\hat{\mathbf{X}}$ can be obtained from the population as:

$$P(\hat{\mathbf{X}} = \mathbf{x}_i) = p_i\,, \ i = 1,2,...,m \ . \tag{2.3}$$

Let $\hat{\mathbf{Y}}$ be the output value of the function *f* with input $\hat{\mathbf{X}}$, that is, $\hat{\mathbf{Y}} = f(\hat{\mathbf{X}})$. In accordance with the discrete probability distribution theory, the distribution sequences of responses vector $\hat{\mathbf{Y}}$ can easily be obtained as

$$P(\hat{\mathbf{Y}} = \mathbf{y}_i) = p_i\,, \ i = 1,2,...,m\,, \tag{2.4}$$

where $\mathbf{y}_i = f(\mathbf{x}_i)$ is the function value when the input variable is selected as the *i*-th value $\mathbf{x}_i$, $i = 1,2,...,m$. The mathematical expectation $\boldsymbol{\mu}_{\hat{y}}$ and covariance matrix $\boldsymbol{\Sigma}_{\hat{y}}$ of the discrete output variable $\hat{\mathbf{Y}}$ are then adopted to approximate the true values of the response variable $\mathbf{Y}$ as:

$$E(\mathbf{Y}) \approx \boldsymbol{\mu}_{\hat{y}}\,, \quad \mathrm{Cov}(\mathbf{Y}) \approx \boldsymbol{\Sigma}_{\hat{y}}\ . \tag{2.5}$$

3. Error Propagation on the PPA

(1) Uncertainty on the Boundary of the PPA

The area of the PPA (i.e. the area of ellipse) is determined by the lengths of the semi-major axis (marked as a) and semi-minor axis (marked as b) of the ellipse which can be solved via the defining expression

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=(t_2-t_1-a_{12})v_{12} \quad . \tag{3.1}$$

The boundary of the PPA, which could also be considered as the search range, obviously affects the computational result of the area. When the rigorous constraints of the classical STP are relaxed, the boundary curve of the PPA becomes uncertain. Therefore, it is necessary to study the uncertainty about the boundary of the PPA in (3.1). To utilize the M-D method, we first transform (3.1) into its parametric equation, i.e.

$$\begin{cases} x=(x_1+x_2)/2+a\cos\theta \\ y=(y_1+y_2)/2+b\sin\theta \end{cases}, \quad \theta\in[0,2\pi] \ , \tag{3.2}$$

where the semi-major axis $a=(t_2-t_1-a_{12})v_{12}/2$, the semi-minor axis $b=\sqrt{a^2-c^2}$, and the half of the focal length $c=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}/2$.

Let $\mathbf{x}=(x_1,y_1,x_2,y_2)$ and $\mathbf{y}=(x,y)$ for any fixed parameter $\theta\in[0,2\pi]$. Then the transfer function $\mathbf{y}=f(\mathbf{x})$ can be implicitly described by (3.2). When $\mathbf{\Sigma}_x$ of the input variable $\mathbf{X}$ is given, the approximate covariance matrix $\mathbf{\Sigma}_{\hat{y}}$ of $\mathbf{Y}=f(\mathbf{X})$ can be obtained by (2.5). Accordingly, the confidence region of the PPA can be defined and plotted on the condition of the uncertain anchor points, uncertain time and uncertain velocity, respectively.

Example 3.1 Consider the case in (3.1). Let t_1= 8, t_2=22, a_{12}=4, v_{12}=2, and the corresponding mean and covariance matrix be

$$\boldsymbol{\mu}_x=(-2,2,6,0), \quad \mathbf{\Sigma}_x=\begin{pmatrix} 0.5 & 0.14 & 0 & 0 \\ 0.14 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}.$$

Fig. 3.1(a) is the 95% confidence region of the PPA which has the error covariance matrix without considering the uncertainty on time and velocity. Fig. 3.1(b) is the error band together with the simulated ellipse curves (in red). The 95% confidence region of the PPA with the uncertain time and anchor region as well as the simulation results are plotted in Fig. 3.1(c). The 95% confidence region of the PPA with the uncertain speed and anchor region, as well as the simulation results are plotted in Fig. 3.1(d).

(2) Uncertainty Propagation on the Intersection Points of Two Circles

In general, the boundary of a PPA is elliptic. It is circular when the origin and destination are the same. In this case, the uncertainty on two objects encountering at arbitrary fixed time t is essentially that on the intersection points of two circles given by the equations

$$\begin{cases} g_1(\mathbf{x},\mathbf{y})\equiv(x-x_1)^2+(y-y_1)^2-(t-t_1)^2v_1^2=0 \\ g_2(\mathbf{x},\mathbf{y})\equiv(x-x_2)^2+(y-y_2)^2-(t-t_2)^2v_2^2=0 \end{cases}, \tag{3.3}$$

where $\mathbf{x}=(x_1,y_1,t_1,v_1,x_2,y_2,t_2,v_2)$ and $\mathbf{y}=(x,y)$. When $\mathbf{x}$ is an input variable and $\mathbf{y}$ is an output variable, the implicit function $\mathbf{y}=f(\mathbf{x})$ can be determined by (3.3). In the following, we compare the uncertainty on the intersection points obtained by the M-D method and the Taylor approximation based on the implicit function theory (Kobayashi *et al*., 2011).

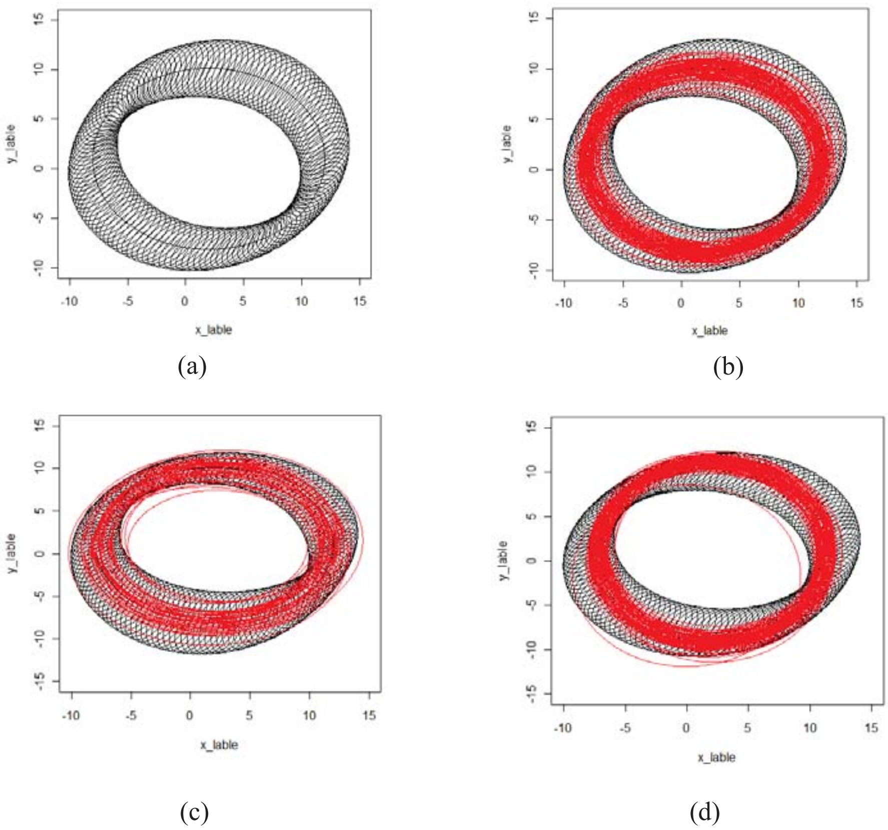

Figure 3.1 The 95% confidence region of the PPA and simulation results

Example 3.2 Let $\boldsymbol{\mu}_x = (2,2,8,2,10,4,20,2)$. Then one of two intersection points for time t =12 is $\boldsymbol{\mu}_y = (\text{-}7.95, 0.84)$, which is given by (3.3). Assume that the covariance matrix $\boldsymbol{\Sigma}_x$ of $\mathbf{X}$ in (2.2) is

$$\boldsymbol{\Sigma}_x = \begin{pmatrix} 0.80 & 0.64 & 0.0 & 0.0 & 0.7 & 0.2 & 0.0 & 0.5 \\ 0.64 & 0.90 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 \\ 0.00 & 0.20 & 0.7 & 0.0 & 0.0 & 0.0 & 0.2 & 0.1 \\ 0.00 & 0.00 & 0.0 & 0.9 & 0.1 & 0.0 & 0.0 & 0.3 \\ 0.70 & 0.10 & 0.0 & 0.1 & 0.6 & 0.0 & 0.1 & 0.0 \\ 0.20 & 0.00 & 0.0 & 0.0 & 0.0 & 0.5 & 0.2 & 0.3 \\ 0.00 & 0.10 & 0.2 & 0.0 & 0.1 & 0.2 & 0.5 & 0.0 \\ 0.50 & 0.00 & 0.1 & 0.3 & 0.0 & 0.3 & 0.0 & 0.2 \end{pmatrix} . \qquad (3.4)$$

The uncertainty on the intersection point $\boldsymbol{\mu}_y$ can be analyzed by the following methods:

a) Taylor approximation based on the implicit function:

$$\mathbf{\Sigma}_y \approx \begin{pmatrix} 126.5308 & -809.0057 \\ -809.0057 & 5467.0158 \end{pmatrix}; \tag{3.5}$$

b) The M-D method:

$$\mathbf{\Sigma}_y \approx \begin{pmatrix} 0.0380219 & -0.02017454 \\ -0.0217454 & 0.01240867 \end{pmatrix}; \tag{3.6}$$

c) Monte Carlo simulation:

We choose 100,000 as the size of the sample on **X**, and find that the effective size of the sample resulted in intersection points is 55422. According to them, we can obtain

$$\mathbf{\Sigma}_y \approx \begin{pmatrix} 23.6009787 & 0.1266690 \\ 0.1266690 & 3.8369359 \end{pmatrix}. \tag{3.7}$$

In contrast with (3.7), it can easily be observed that the M-D results in (3.6) is superior to that in (3.5) since the maximum relative error of (3.6) is about one while that of (3.5) is too large to be of value. It is apparent that when the transfer function is highly nonlinear, error propagation analysis based on the M-D method is more feasible and effective than that on the Taylor approximation of the implicit function.

4. Conclusion

We have investigated in this paper the propagation of uncertainty in space-time prisms. In particular, the propagation of error of the PPA and its intersections has been analyzed via the moment-design method. It has been shown that it is an appealing and feasible method for error propagation analysis under certain conditions, especially nonlinearity. Although the method can effectively capture the nonlinearity of the transfer function to a certain extent and gives more precise results than the implicit function approach, there is still a difference between results obtained by the M-D method and Monte Carlo method. It is thus necessary to carry out further study of the M-D method for error propagation problems.

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